

ON p -ADIC SOLID-ON-SOLID MODEL ON A CAYLEY TREE

O. N. KHAKIMOV

ABSTRACT. We consider a nearest-neighbor p -adic Solid-on-Solid (SOS) model with $m+1$ spin values and coupling constant $J \in \mathbb{Q}_p$ on a Cayley tree. It is found conditions under which a phase transition does not occur for this model. It is shown that under condition $p \mid m+1$ for some J a phase transition occurs. Moreover, we give criterion of boundedness of p -adic Gibbs measures for the $m+1$ -state SOS model.

Key words. p -adic number, p -adic SOS model, Cayley tree, p -adic Gibbs measure.

1. INTRODUCTION

The p -adic numbers were first introduced by the German mathematician K.Hensel. They form an integral part of number theory, algebraic geometry, representation theory and other branches of modern mathematics. However, numerous applications of these numbers to theoretical physics have been proposed [1] to quantum mechanics and to p -adic valued physical observable [6]. A number of p -adic models in physics cannot be described using ordinary probability theory based on the Kolmogorov axioms.

In [9] a theory of stochastic processes with values in p -adic and more general non-Archimedean fields was developed, having probability distributions with non-Archimedean values.

One of the basic branches of mathematics lying at the base of the theory of statistical mechanics is the theory of probability and stochastic processes. Since the theories of probability and stochastic processes in a non-Archimedean setting have been introduced, it is natural to study problems of statistical mechanics in the context of the p -adic theory of probability.

We note that p -adic Gibbs measures were studied for several p -adic models of statistical mechanics [2, 3, 5], [10–13].

The SOS model can be considered as a generalization of the Ising model (which arises when $m = 1$). It is known that a phase transition does not occur for the p -adic Ising model on a Cayley tree. We prove that there is no phase transition for the $m+1$ -state p -adic SOS model if $p \nmid m+1$. Moreover, we show that if $p \mid m+1$ then a phase transition may occur for this model.

The organization of this paper as follows. Section 2 is a mathematically preliminary. In section 3 we give a construction of p -adic Gibbs measures for the $m+1$ -state p -adic SOS model on a Cayley tree of order $k \geq 1$. Moreover, we study a problem of boundedness of such measures. In section 4 we study the set of all translation-invariant

p -adic Gibbs measures for 3-state p -adic SOS model. In section 5 we give a criterion of uniqueness of the p -adic Gibbs measure.

2. DEFINITIONS AND PRELIMINARY RESULTS

2.1. p -adic numbers and measures. Let \mathbb{Q} be the field of rational numbers. For a fixed prime number p , every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p : $(p, n) = 1$, $(p, m) = 1$. The p -adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

This norm is non-Archimedean and satisfies the so called strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

From this property immediately follow the following facts (*non-Archimedean norm's property*):

- 1) if $|x|_p \neq |y|_p$, then $|x - y|_p = \max\{|x|_p, |y|_p\}$;
- 2) if $|x|_p = |y|_p$, then $|x - y|_p \leq |x|_p$;

The completion of \mathbb{Q} with respect to the p -adic norm defines the p -adic field \mathbb{Q}_p (see [8]).

The completion of the field of rational numbers \mathbb{Q} is either the field of real numbers \mathbb{R} or one of the fields of p -adic numbers \mathbb{Q}_p (Ostrowski's theorem).

Any p -adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots), \quad (2.1)$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 > 0$, $0 \leq x_j \leq p - 1$ (see [8, 14, 15]). In this case $|x|_p = p^{-\gamma(x)}$.

Theorem 1. [15] *The equation $x^2 = a$, $0 \neq a = p^{\gamma(a)}(a_0 + a_1p + \dots)$, $0 \leq a_j \leq p - 1$, $a_0 > 0$ has a solution $x \in \mathbb{Q}_p$ iff hold true the following:*

- (a) $\gamma(a)$ is even;
- (b) $y^2 \equiv a_0 \pmod{p}$ is solvable for $p \neq 2$; the equality $a_1 = a_2 = 0$ holds if $p = 2$.

We respectively denote the sets of all p -adic integers and units of \mathbb{Q}_p by

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}, \quad \mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}.$$

Lemma 1. (Hensel's lemma, [15]). *Let $f(x)$ be a polynomial whose the coefficients are p -adic integers. Let a_0 be a p -adic integer such that for some $i \geq 0$ we have*

$$\begin{aligned} f(a_0) &\equiv 0 \pmod{p^{2i+1}}, \\ f'(a_0) &\equiv 0 \pmod{p^i}, \quad f'(a_0) \not\equiv 0 \pmod{p^{i+1}}. \end{aligned}$$

Then $f(x)$ has a unique p -adic integer root x_0 which satisfies $x_0 \equiv a_0 \pmod{p^{i+1}}$.

For $a \in \mathbb{Q}_p$ and $r > 0$ we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$

p -adic *logarithm* is defined by the series

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n},$$

which converges for $x \in B(1, 1)$; p -adic *exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Lemma 2. *Let $x \in B(0, p^{-1/(p-1)})$. Then*

$$\begin{aligned} |\exp_p(x)|_p &= 1, \quad |\exp_p(x) - 1|_p = |x|_p, \quad |\log_p(1 + x)|_p = |x|_p, \\ \log_p(\exp_p(x)) &= x, \quad \exp_p(\log_p(1 + x)) = 1 + x. \end{aligned}$$

Denote

$$\mathcal{E}_p = \left\{ x \in \mathbb{Q}_p : |x|_p = 1, |x - 1|_p < p^{-1/(p-1)} \right\}.$$

A more detailed description of p -adic calculus and p -adic mathematical physics can be found in [8, 14, 15].

Let (X, \mathcal{B}) be a measurable space, where \mathcal{B} is an algebra of subsets of X . A function $\mu : \mathcal{B} \rightarrow \mathbb{Q}_p$ is said to be a p -adic measure if for any $A_1, \dots, A_n \in \mathcal{B}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$, the following holds:

$$\mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j).$$

A p -adic measure is called a probability measure if $\mu(X) = 1$. A p -adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$ (see [6]).

We call a p -adic measure a probability measure [3] if $\mu(X) = 1$.

2.2. Cayley tree. The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k + 1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them. We shall use the notation $l = \langle x, y \rangle$. A collection of nearest neighbor pairs $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from x to y . The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y .

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^n W_m$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

the set of *direct successors* of x .

Let G_k be a free product of $k + 1$ cyclic groups of the second order with generators a_1, a_2, \dots, a_{k+1} , respectively. It is known that there exists a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k .

2.3. p -adic SOS model. Let \mathbb{Q}_p be the field of p -adic numbers and Φ be a finite set. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in \Phi$; in a similar fashion one defines a configuration σ_n and $\sigma^{(n)}$ on V_n and W_n respectively. The set of all configurations on V (resp. V_n , W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \omega(x), & \text{if } x \in W_n. \end{cases}$$

It is clear that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

Let G_k^* be a subgroup of the group G_k . A function h_x (for example, a configuration $\sigma(x)$) of $x \in G_k$ is called G_k^* -periodic if $h_{yx} = h_x$ (resp. $\sigma(yx) = \sigma(x)$) for any $x \in G_k$ and $y \in G_k^*$.

A G_k -periodic function is called *translation-invariant*.

We consider p -adic SOS model on a Cayley tree, where the spin takes values in the set $\Phi := \{1, 2, \dots, m\}$, and is assigned to the vertices of the tree.

The (formal) Hamiltonian is of a p -adic SOS form:

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} |\sigma(x) - \sigma(y)|_\infty, \quad (2.2)$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant, $\langle x, y \rangle$ stands for nearest neighbor vertices and $|\cdot|_\infty$ stands for usual absolute value.

The p -adic SOS model of this type can be considered as a generalization of the p -adic Ising model (which arises when $m = 1$).

3. THE SYSTEM OF p -ADIC VECTOR-VALUED FUNCTIONAL EQUATIONS

We use standard definition of a p -adic Gibbs measure, a translation-invariant (TI) measure. Also, call measure μ *symmetric* if it is preserved under simultaneous change $j \mapsto m - j$ at each vertex $x \in V$.

Let $z : x \mapsto z_x = (z_{0,x}, z_{1,x}, \dots, z_{m,x}) \in \mathcal{E}_p^{m+1}$ be a p -adic vector-valued function of $x \in V \setminus \{x_0\}$. Given $n = 1, 2, \dots$, consider the p -adic probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu_{\tilde{z}}^{(n)}(\sigma_n) = Z_{n,\tilde{z}}^{-1} \exp_p \{H_n(\sigma_n)\} \prod_{x \in W_n} \tilde{z}_{\sigma(x),x}. \quad (3.1)$$

Here, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ and $Z_{n,\tilde{z}}$ is the corresponding partition function:

$$Z_{n,\tilde{z}} = \sum_{\sigma \in \Omega_{V_n}} \exp_p \{H_n(\sigma_n)\} \prod_{x \in W_n} \tilde{z}_{\sigma(x),x}. \quad (3.2)$$

We say that the p -adic probability distributions (3.1) are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n \in \Omega_{W_n}} \mu_{\tilde{z}}^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu_{\tilde{z}}^{(n-1)}(\sigma_{n-1}). \quad (3.3)$$

Here $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations.

We note that an analog of the Kolmogorov extension theorem for distributions can be proved for p -adic distributions given by (3.1) (see [3]). According to this theorem there exists a unique p -adic measure $\mu_{\tilde{z}}$ on Ω such that, for all n and $\sigma_n \in \Omega_{V_n}$,

$$\mu_{\tilde{z}}(\{\sigma|_{V_n} = \sigma_n\}) = \mu_{\tilde{z}}^{(n)}(\sigma_n).$$

Such a measure is called a p -adic Gibbs measure (pGM) corresponding to the Hamiltonian (2.2) and vector-valued function $\tilde{z}_x, x \in V$. We denote by $G(H)$ the set of all p -adic Gibbs measures for the hamiltonian H . If $|G(H)| \geq 2$ then we say that for this model there is a *phase transition*.

The following statement describes conditions on \tilde{z}_x guaranteeing compatibility of $\mu_{\tilde{z}}^{(n)}(\sigma_n)$.

Proposition 1. *The p -adic probability distributions $\mu_{\tilde{z}}^{(n)}(\sigma_n)$, $n = 1, 2, \dots$ in (3.1) are compatible for p -adic SOS model iff for any $x \in V \setminus \{x^0\}$ the following system of equations holds:*

$$z_{i,x} = \prod_{y \in S(x)} \frac{\sum_{j=0}^{m-1} \theta^{|i-j|_\infty} z_{j,y} + \theta^{m-i}}{\sum_{j=0}^{m-1} \theta^{m-j} z_{j,y} + 1}, \quad i = 0, 1, \dots, m-1. \quad (3.4)$$

Here $\theta = \exp_p(J)$ and $z_{i,x} = \frac{\tilde{z}_{i,x}}{\tilde{z}_{m,x}}$, $i = 0, 1, \dots, m-1$.

Proof. Necessity. Suppose that (3.3) holds; we want to prove (3.4). Substituting (3.1) in (3.3), obtain, for any configurations $\sigma_{n-1} \in \Omega_{V_n}$:

$$\frac{Z_{n-1,\tilde{z}}}{Z_{n,\tilde{z}}} \sum_{\omega \in \Omega_{W_n}} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \exp_p(J|\sigma_{n-1}(x) - \omega(y)|_\infty) \tilde{z}_{\omega(y),y} = \prod_{x \in W_{n-1}} \tilde{z}_{\sigma_{n-1}(x),x}.$$

From this for any $i \in \{0, 1, \dots, m\}$ we get

$$\prod_{y \in S(x)} \frac{\sum_{j=0}^m \exp_p(J|i-j|_\infty) \tilde{z}_{j,y}}{\sum_{j=0}^m \exp_p(J(m-j)) \tilde{z}_{j,y}} = \frac{\tilde{z}_{i,x}}{\tilde{z}_{m,x}}. \quad (3.5)$$

Denoting $\theta = \exp_p(J)$ and $z_{i,x} = \frac{\tilde{z}_{i,x}}{\tilde{z}_{m,x}}$, we get (3.4) from (3.5).

Sufficiency. Suppose that (3.4) holds. It is equivalent to the representations

$$\prod_{y \in S(x)} \sum_{j=0}^m \theta^{|i-j|_\infty} \tilde{z}_{j,y} = a(x) \tilde{z}_{i,x}, \quad i = 0, 1, \dots, m-1 \quad (3.6)$$

for some function $a(x)$. We have

$$\text{LHS of (3.3)} = \frac{1}{Z_{n,\tilde{z}}} \exp_p(H(\sigma_{n-1})) \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \sum_{j=0}^m \theta^{|i-j|_\infty} \tilde{z}_{i,y}. \quad (3.7)$$

Substituting (3.6) into (3.7) and denoting $A_n(x) = \prod_{y \in W_{n-1}} a(y)$, we get

$$\text{RHS of (3.7)} = \frac{A_{n-1}}{Z_{n,\tilde{z}}} \exp_p(H(\sigma_{n-1})) \prod_{x \in W_{n-1}} \tilde{z}_{\sigma_{n-1}(x),x}. \quad (3.8)$$

Since $\mu_{\tilde{z}}^{(n)}$, $n \geq 1$ is a probability, we should have

$$\sum_{\sigma_{n-1} \in \Omega_{V_{n-1}}} \sum_{\omega \in \Omega_{W_n}} \mu_{\tilde{z}}^{(n)}(\sigma_{n-1}, \omega) = 1.$$

Hence from (3.8) we get $Z_{n,\tilde{z}} = A_{n-1} Z_{n-1,\tilde{z}}$, and (3.3) holds. \square

The following proposition is straightforward.

Proposition 2. *Let H be a hamiltonian of $m+1$ -state p -adic SOS model on a Cayley tree Γ^k . Then it hold the following statements:*

- 1) *Any measure μ with local distributions $\mu^{(n)}$ satisfying (3.1), (3.3) belongs to $G(H)$.*
- 2) *An $\mu \in G(H)$ is TI iff $z_{i,x}$ does not depend on $x : z_{i,x} \equiv z_i$, $x \in V$, $i \in \Phi$, and symmetric TI if and only if $z_i = z_{m-i}$, $i \in \Phi$.*

3.1. Boundedness of p -adic Gibbs measure.

Theorem 2. *Let H be a $m+1$ -state p -adic SOS model on a Cayley tree of order k . Then a measure $\mu \in G(H)$ is bounded if and only if $m+1$ is not divisible by p .*

Proof. Let $z_x = (z_{0,x}, z_{1,x}, \dots, z_{m-1,x})$ is a solution to (3.4). Then from (3.6) for all $x \in V \setminus \{x_0\}$ we find

$$a(x) = \prod_{y \in S(x)} \left(\sum_{j=0}^{m-1} \theta^{m-j} z_{j,y} + 1 \right) = \prod_{y \in S(x)} \left(\sum_{j=0}^{m-1} (\theta^{m-j} z_{j,y} - 1) + m + 1 \right).$$

Since $\theta \in \mathcal{E}_p$ and $z_x \in \mathcal{E}_p^m$, by non-Archimedean norm's property we get

$$|a(x)|_p = \begin{cases} 1, & \text{if } p \nmid m+1; \\ \leq p^{-k}, & \text{if } p \mid m+1. \end{cases}$$

From this we obtain

$$|A_n(x)|_p = \prod_{y \in W_{n-1}} |a(y)|_p = \begin{cases} 1, & \text{if } p \nmid m+1; \\ \leq p^{-k|W_{n-1}|}, & \text{if } p \mid m+1. \end{cases} \quad (3.9)$$

We use the following recurrence formula

$$Z_{n,z} = A_{n-1} Z_{n-1,z}.$$

From (3.9) we get

$$|Z_{n,z}|_p = \prod_{x \in V_{n-1}} |A_{n-1}(x)|_p = \begin{cases} 1, & \text{if } p \nmid m+1; \\ \leq p^{-k|V_{n-1}|}, & \text{if } p \mid m+1. \end{cases} \quad (3.10)$$

For any configuration $\sigma \in \Omega_{V_n}$ by (3.10) we have

$$\begin{aligned} \left| \mu_z^{(n)}(\sigma) \right|_p &= \frac{|\exp_p\{H_n(\sigma)\} \prod_{x \in W_n} z_{\sigma(x),x}|_p}{|Z_{n,z}|_p} = \\ \frac{1}{|Z_{n,z}|_p} &= \begin{cases} 1, & \text{if } p \nmid m+1; \\ \geq p^{k|V_{n-1}|}, & \text{if } p \mid m+1. \end{cases} \end{aligned}$$

which means the measure μ_z is bounded if and only if $p \nmid m+1$. \square

4. THREE-STATE SOS MODEL

From Proposition 2 it follows that for any $z = \{z_x, x \in V\}$ satisfying (3.4) there exists a unique p -adic Gibbs measure μ . Denote by TIpGM the set of all translation-invariant p -adic Gibbs measures for the model (2.2). Note that $\text{TIpGM} \subset G(H)$. However, description of the set TIpGM for an arbitrary m is not easy. We now suppose that $m = 2$, i.e. $\Phi = \{0, 1, 2\}$. We assume that $z_{2,x} \equiv 1$ ($z_{m,x} \equiv 1$ for general m).

4.1. Translation-invariant solutions. It is natural begin with TI solutions where $z_x = z \in \mathcal{E}_p^m$ is constant. Unless otherwise stated, we concentrate on the simplest case where $m = 2$, i.e., spin values are 0, 1 and 2. In this case we obtain from (3.4):

$$z_0 = \left(\frac{z_0 + \theta z_1 + \theta^2}{\theta^2 z_0 + \theta z_1 + 1} \right)^k, \quad (4.1)$$

$$z_1 = \left(\frac{\theta z_0 + z_1 + \theta}{\theta^2 z_0 + \theta z_1 + 1} \right)^k. \quad (4.2)$$

Proposition 3. *If $p \neq 3$ then the system of equations (4.1), (4.2) has no solution in $\mathcal{A}_p = \{(z_0, z_1) \in \mathcal{E}_p^2 : z_0 \neq 1\}$.*

Proof. Let $p \neq 3$. Denote $A = z_0 + \theta z_1 + \theta^2$, $B = \theta^2 z_0 + \theta z_1 + 1$. Then from (4.1) we get:

$$(z_0 - 1) \left(B^k + (\theta^2 - 1) \sum_{i=0}^{k-1} A^{k-1-i} B^i \right) = 0. \quad (4.3)$$

Since $\theta, z_0, z_1 \in \mathcal{E}_p$ we have

$$A \equiv 3 \pmod{p} \quad \text{and} \quad B \equiv 3 \pmod{p}.$$

Consequently,

$$\left| B^k + (\theta^2 - 1) \sum_{i=0}^{k-1} A^{k-1-i} B^i \right|_p = |3^k|_p = 1. \quad (4.4)$$

From (4.3) and (4.4) we obtain $z_0 = 1$. \square

Observe that $z_0 = 1$ satisfies equation (4.1) independently of k, θ and z_1 . Substituting $z_0 = 1$ into (4.2), we have

$$z_1 = \left(\frac{2\theta + z_1}{\theta^2 + \theta z_1 + 1} \right)^k. \quad (4.5)$$

We consider the function

$$f(x) = \left(\frac{2a + x}{a^2 + ax + 1} \right)^k.$$

Let $p \neq 3$ and $a \in \mathcal{E}_p$. Then for any $x \in \mathcal{E}_p$ from non-Archimedean norm's property we get

$$|f(x)|_p = \left(\frac{|2a + x|_p}{|a^2 + ax + 1|_p} \right)^k = 1,$$

and

$$|f(x) - 1|_p = \left| \frac{(a-1)(1-x-a) \sum_{i=0}^{k-1} (2a+x)^{k-1-i} (a^2+ax+1)^i}{(a^2+ax+1)^k} \right|_p < p^{-1/(p-1)}.$$

Thus, we have shown that $f : \mathcal{E}_p \mapsto \mathcal{E}_p$ if $p \neq 3$ and $a \in \mathcal{E}_p$.

Now we shall show that $|f'(x)|_p < 1$ for any $x \in \mathcal{E}_p$.

$$f'(x) = -\frac{k(a^2-1)}{(a^2+ax+1)^2} \left(\frac{2a+x}{a^2+ax+1} \right)^{k-1}.$$

Since $p \neq 3$ and $a \in \mathcal{E}_p$ we obtain

$$|f'(x)|_p = |k(a^2-1)|_p \leq \frac{1}{p}.$$

Hence, we get

$$|f(x) - f(y)|_p \leq \frac{1}{p} |x - y|_p \quad \text{for all } x, y \in \mathcal{E}_p.$$

Consequently, the function f has a unique fixed point x^* as \mathcal{E}_p is compact. Thus, we have proved the following proposition

Proposition 4. *If $p \neq 3$ then the system of equations (4.1), (4.2) has a unique solution in $\mathcal{E}_p^2 \setminus \mathcal{A}_p$.*

From Proposition 3 and Proposition 4 we get the following

Theorem 3. *Let $p \neq 3$. Then there exists a unique translation-invariant p -adic Gibbs measure for the three-state p -adic SOS model on a Cayley tree of order k .*

Proposition 5. *Let $p = 3$. If $\theta \in \{x \in \mathcal{E}_3 : |x - 1|_3 < 3^{-2}\}$ then the system of equations (4.1), (4.2) has a solution in \mathcal{E}_3^2 .*

Proof. Let $p = 3$. We consider the following polynomial with p -adic integers coefficients:

$$g(x) = \theta x^{k+1} - x^k + (\theta^2 + 1)x - 2\theta. \quad (4.6)$$

We will check conditions of Hensel's lemma.

$$g(1) = \theta(\theta - 1) \quad \text{and} \quad g'(1) = (\theta - 1)(\theta + k + 2) + 3. \quad (4.7)$$

Since $\theta \in \{x \in \mathcal{E}_3 : |x - 1|_3 < 3^{-2}\}$ we get

$$\theta \equiv 1 \pmod{27}. \quad (4.8)$$

Using this, from (4.7) we have

$$\begin{aligned} g(1) &\equiv 0 \pmod{27}, \\ g'(1) &\equiv 0 \pmod{3}, \quad g'(1) \not\equiv 0 \pmod{9}. \end{aligned}$$

Thus, we have shown that for the function $g(x)$ all conditions of Hensel's lemma are satisfied. Then the function $g(x)$ has a unique p -adic integer root x_* which satisfies $x_* \equiv 1 \pmod{9}$.

It is easy to see $x_*^k \in \mathcal{E}_3$ for any $k \geq 1$ and $z_1^* = x_*^k$ is a solution to (4.5). Consequently, $(1, z_1^*) \in \mathcal{E}_3^2$ is a solution to (4.1), (4.2). \square

Proposition 6. *Let $p = 3$. If k is an even and it is not divisible by p then a system of equations (4.1), (4.2) has a solution in \mathcal{E}_3^2 .*

Proof. Let k is an even and it is not divisible by 3. Then we get

$$2^k \equiv 1 \pmod{3} \quad \text{and} \quad k \not\equiv 0 \pmod{3}. \quad (4.9)$$

Consider the function $g(x)$ (see (4.6)). We have

$$g(2) = 2^{k+1}\theta - 2^k + 2(\theta^2 - \theta + 1)$$

and

$$g'(2) = (k+1)2^k\theta - k2^{k-1} + \theta^2 + 1.$$

Since $\theta \in \mathcal{E}_3$ and using (4.9), we obtain

$$g(2) = 2^{k+1}\theta - 2^k + 2(\theta^2 - \theta + 1) \equiv 0 \pmod{3},$$

and

$$g'(2) = (k+1)2^k\theta - k2^{k-1} + \theta^2 + 1 \not\equiv 0 \pmod{3}.$$

Then by Hensel's lemma there exists a unique p -adic integer \tilde{x}_* such that $g(\tilde{x}_*) = 0$ and $\tilde{x}_* \equiv 2 \pmod{3}$. It is easy to check that $\tilde{x}_*^k \in \mathcal{E}_3$. Denote $\tilde{z}_* = \tilde{x}_*^k$. Then \tilde{z}_* is a solution to (4.2). Consequently, $(1, \tilde{z}_*) \in \mathcal{E}_3^2$ is a solution to (4.1), (4.2). \square

Corollary 1. *Let $p = 3$ and $\theta \in \{x \in \mathcal{E}_3 : |x - 1|_3 < 3^{-2}\}$. If k is an even and it is not divisible by p then a system of equations (4.1), (4.2) has at least two solutions in \mathcal{E}_3^2 .*

Theorem 4. *Let $p = 3$ and $\theta \in \{x \in \mathcal{E}_3 : |x - 1|_3 < 3^{-2}\}$. If k is an even and it is not divisible by p then a phase transition occurs for the three state p -adic SOS model on a Cayley tree of order k .*

Proposition 7. *Let $p = 3$ and $k = 2$.*

1) *If $\theta \notin \{x \in \mathcal{E}_3 : |x - 10|_3 < 3^{-2}\}$ then the system of equations (4.1), (4.2) has no solutions in $\mathcal{A}_3 = \{(z_0, z_1) \in \mathcal{E}_3^2 : z_0 \neq 1\}$.*

2) *If $\theta \in \{x \in \mathcal{E}_3 : |x - 37|_3 < 3^{-3}\}$ then the system of equations (4.1), (4.2) has two solutions in $\mathcal{A}_3 = \{(z_0, z_1) \in \mathcal{E}_3^2 : z_0 \neq 1\}$.*

Proof. Let $p = 3$ and $k = 2$. Since $z_i \in \mathcal{E}_p$, $i = 1, 2$ by Theorem 1 there exist $\sqrt{z_i}$ in \mathbb{Q}_p . Denote $x = \sqrt{z_0}$ and $y = \sqrt{z_1}$. Rewrite (4.1) and (4.2) as

$$x = \frac{x^2 + \theta y^2 + \theta^2}{\theta^2 x^2 + \theta y^2 + 1}, \quad (4.10)$$

$$y = \frac{\theta x^2 + y^2 + \theta}{\theta^2 x^2 + \theta y^2 + 1}. \quad (4.11)$$

From (4.10) we get

$$(x - 1)(\theta^2 x^2 + \theta y^2 + 1 + (\theta^2 - 1)(x + 1)) = 0. \quad (4.12)$$

Let $x \neq 1$. Then from (4.12) we have

$$\theta^2 x^2 + \theta y^2 + 1 = (1 - \theta^2)(x + 1). \quad (4.13)$$

Put this to (4.11) and obtain

$$\theta y = \frac{x}{x + 1}. \quad (4.14)$$

Hence we conclude that

$$x \equiv 1 \pmod{3} \quad \text{and} \quad y \equiv 2 \pmod{3}.$$

From (4.13) and (4.14) we get

$$\theta^3 x^4 + \theta(3\theta^2 - 1)x^3 + (4\theta^3 - 2\theta + 1)x^2 + \theta(3\theta^2 - 1)x + \theta^3 = 0. \quad (4.15)$$

Note that $x = 1$ is a solution to (4.15) if and only if $12\theta^3 - 4\theta + 1 = 0$.

Let $12\theta^3 - 4\theta + 1 = 0$. Then from (4.15) we get

$$(x - 1)^2 (\theta^3 x^2 + \theta(5\theta^2 - 1)x + \theta^3) = 0.$$

Consider the following equation

$$\theta^3 x^2 + \theta(5\theta^2 - 1)x + \theta^3 = 0. \quad (4.16)$$

This equation is solvable in \mathbb{Q}_3 if and only if $\sqrt{1 - 7\theta^2}$ exists in \mathbb{Q}_3 . We will show that the quadratic equation (4.16) is not solvable in \mathbb{Q}_3 . From $12\theta^3 - 4\theta + 1 = 0$ and $\theta \in \mathcal{E}_3$ we get $\theta \equiv 10 \pmod{27}$. Then we have

$$1 - 7\theta^2 \equiv 3 \pmod{27}.$$

Hence by Theorem 1 there does not exist $\sqrt{1 - 7\theta^2}$ in \mathbb{Q}_3 . Consequently, the equation (4.16) is not solvable in \mathbb{Q}_3 .

Let $12\theta^3 - 4\theta + 1 \neq 0$. Denote $t = x + x^{-1}$. Then from (4.15) we get

$$\theta^3 t^2 + \theta(3\theta^2 - 1)t + 2\theta^3 - 2\theta + 1 = 0. \quad (4.17)$$

Denote $D = \theta^4 + 2\theta^2 - 4\theta + 1$. The equation (4.17) has two distinct solutions if there exists \sqrt{D} in \mathbb{Q}_3 . Moreover, we have $t_i \equiv 2 \pmod{3}$, $i = 1, 2$.

The solutions of the equation (4.15) are

$$x^\pm = \frac{1 - 3\theta^2 + \sqrt{D} \pm \sqrt{(1 - 7\theta^2 + \sqrt{D})(1 + \theta^2 + \sqrt{D})}}{4\theta^2},$$

$$x_\pm = \frac{1 - 3\theta^2 - \sqrt{D} \pm \sqrt{(1 - 7\theta^2 - \sqrt{D})(1 + \theta^2 - \sqrt{D})}}{4\theta^2}.$$

Note that the existence of solutions x^\pm equivalent to the existence of \sqrt{D} and the existence of $\sqrt{2(1 - 7\theta^2 - \sqrt{D})}$ in \mathbb{Q}_3 . Moreover, if the solutions x^\pm exists then $x^\pm \equiv 1 \pmod{3}$.

We will show that the number $\sqrt{2(1 - 7\theta^2 + \sqrt{D})}$ does not exist in \mathbb{Q}_3 . At first we will check the existence of \sqrt{D} . We have

$$D = \theta^4 + 2\theta^2 - 4\theta + 1 = (\theta - 1)(4\theta + (\theta - 1)(\theta + 1)^2).$$

Since $\theta \in \mathcal{E}_3$ by Theorem 1 we conclude that the \sqrt{D} exists if and only if

$$\theta = 1 + 3^{2n}(1 + \varepsilon), \quad n \in \mathbb{N}, \quad |\varepsilon|_3 < 1. \quad (4.18)$$

Then we get

$$\sqrt{D} = 3^n(1 + \varepsilon'), \quad |\varepsilon'|_3 < 1.$$

Hence, for all $n \in \mathbb{N}$ we have

$$\begin{aligned} |1 - 7\theta^2 + \sqrt{D}|_3 &= |1 - 7(1 + 3^{2n}(1 + \varepsilon)) + 3^n(1 + \varepsilon')|_3 = \\ &= |-6 + 3^n + 3^n\varepsilon' - 14(1 + \varepsilon)3^{2n} + 7(1 + \varepsilon)^2 3^{4n}|_3 = \frac{1}{3}. \end{aligned} \quad (4.19)$$

By Theorem 1 the number $\sqrt{2(1 - 7\theta^2 + \sqrt{D})}$ does not exist in \mathbb{Q}_3 .

Now we will check the existence of $\sqrt{2(1 - 7\theta^2 - \sqrt{D})}$. Using (4.18) we obtain

$$|1 - 7\theta^2 + \sqrt{D}|_3 = \begin{cases} \frac{1}{3}, & \text{if } n > 1; \\ < \frac{1}{3}, & \text{if } n = 1. \end{cases}$$

Hence, by Theorem 1 there does not exist $\sqrt{2(1 - 7\theta^2 - \sqrt{D})}$ in \mathbb{Q}_3 if $n > 1$. So, we must check the case $n = 1$.

Let $|\theta - 37|_3 < 3^{-3}$, i.e., $\theta = 37 + \beta$, $|\beta|_3 < 3^{-3}$. Then we obtain

$$(1 - 7\theta^2 + \sqrt{D})(1 - 7\theta^2 - \sqrt{D}) = 4\theta(12\theta^3 - 4\theta + 1) =$$

$$4(37 + \beta) (12(37 + \beta)^3 - 4(37 + \beta) + 1) = 27(1 + 2 \cdot 3 + 3^2 + \dots) \quad (4.20)$$

On the other hand by (4.19) one can find

$$1 - 7\theta^2 + \sqrt{D} = 6(1 + \alpha_1 \cdot 3 + \alpha_2 \cdot 3^2 + \dots), \quad \alpha_i \in \{0, 1, 2\}, \quad i = 1, 2, \dots$$

From this and by (4.20) we get

$$2(1 - 7\theta^2 - \sqrt{D}) = 9(1 + \gamma_1 \cdot 3 + \gamma_2 \cdot 3^2 + \dots), \quad \gamma_i \in \{0, 1, 2\}, \quad i = 1, 2, \dots$$

Then by Theorem 1 there exists $\sqrt{2(1 - 7\theta^2 - \sqrt{D})}$ in \mathbb{Q}_3 . Hence, (x_+, y_+) and (x_-, y_-) are the solutions to (4.10), (4.11), where $y_{\pm} = \frac{x_{\pm}}{\theta(x_{\pm} + 1)}$. Redenote $z_0^{\pm} = x_{\pm}$ and $z_1^{\pm} = y_{\pm}$. It is clear that $z_i^{\pm} \in \mathcal{E}_3$, $i = 0, 1$ and $z_0^{\pm} \neq 1$.

Thus, we have shown that the system of equations (4.1), (4.2) has two solutions in \mathcal{A}_3 if $|\theta - 37|_3 < 3^{-3}$ and it has no solution in \mathcal{A}_3 if $|\theta - 1|_3 < 3^{-2}$. \square

Thus by Proposition 7 we have

Theorem 5. *A phase transition occurs for the three state 3-adic SOS model on a Cayley tree of order two if one of the following statements hold:*

- 1) $\theta \in \{x \in \mathcal{E}_3 : |x - 1|_3 < 3^{-2}\}$;
- 2) $\theta \in \{x \in \mathcal{E}_3 : |x - 37|_3 < 3^{-3}\}$.

5. THE UNIQUENESS OF p -ADIC GIBBS MEASURES

In the previous section we have shown that if $p \neq 3$ then there is no phase transition for the three state p -adic SOS model. A natural question arises: what should be the relation between a number m and prime p in order to have a phase transition for the $m + 1$ -state p -adic SOS model? In this section we shall find this relation.

Let us first prove some technical results.

Lemma 3. *[10] If $a_i \in \mathcal{E}_p$, $i = 1, 2, \dots, n$ then $\prod_{i=1}^n a_i \in \mathcal{E}_p$.*

Recall that the p -adic norm of $x \in \mathbb{Q}_p^m$ defined as

$$\|x\|_p = \max_{1 \leq i \leq m} \{|x_i|_p\}.$$

Let the collection of functions $F_i : \mathbb{Q}_p^{m+1} \mapsto \mathbb{Q}_p$, $i = 0, 1, \dots, m$ given by

$$F_i(z, a_i, b_i, m) = \frac{\sum_{j=0}^m a_{ij} z_j}{\sum_{j=0}^m b_{ij} z_j},$$

where

$$a_i = (a_{i0}, a_{i1}, \dots, a_{im}) \in \mathcal{E}_p^{m+1} \quad \text{and} \quad b_i = (b_{i0}, b_{i1}, \dots, b_{im}) \in \mathcal{E}_p^{m+1},$$

for any $i = 0, 1, \dots, m$. For convenience we write $F_i(z)$ instead of $F_i(z, a_i, b_i, m)$.

Lemma 4. *Let $m + 1$ is not divisible by p . Then F_i is a function from \mathcal{E}_p^{m+1} to \mathcal{E}_p for any $i = 0, 1, \dots, m$.*

Proof. Let $z \in \mathcal{E}_p^{m+1}$. We will show that

$$|F_i(z) - 1|_p < p^{-1/(p-1)}.$$

By Lemma 3 we have $a_{ij}z_j, b_{ij}z_j \in \mathcal{E}_p$ for all $j = 0, 1, \dots, m$ as $a_{ij}, b_{ij}, z_j \in \mathcal{E}_p$. Since $m+1 \not\equiv 0 \pmod{p}$ and using non-Archimedean norm's property we get

$$\left| \sum_{j=0}^m b_{ij}z_j \right|_p = \left| \sum_{j=0}^m (b_{ij}z_j - 1) + m + 1 \right|_p = 1. \quad (5.1)$$

From this we obtain

$$|F_i(z) - 1|_p = \frac{\left| \sum_{j=0}^m (a_{ij} - b_{ij})z_j \right|_p}{\left| \sum_{j=0}^m b_{ij}z_j \right|_p} \leq \max_{0 \leq j \leq m} \{|a_{ij} - b_{ij}|_p\} < p^{-1/(p-1)}.$$

□

Lemma 5. *Let $m+1$ is not divisible by p . Then for any $z, t \in \mathcal{E}_p^{m+1}$ it holds*

$$|F_i(z) - F_i(t)|_p \leq \frac{1}{p} \|z - t\|_p, \quad i = 0, 1, \dots, m.$$

Proof. Let $m+1 \not\equiv 0 \pmod{p}$. Then for any $z, t \in \mathcal{E}_p$ from (5.1) we get

$$\begin{aligned} |F_i(z) - F_i(t)|_p &= \frac{\left| \sum_{j=0}^m \sum_{l=0}^m a_{ij}b_{il}(z_jt_l - z_lt_j) \right|_p}{\left| \sum_{j=0}^m a_{ij}z_j \sum_{l=0}^m b_{il}t_l \right|_p} = \\ &= \frac{\left| \sum_{j=0}^m \sum_{l=0}^m a_{ij}b_{il} (t_l(z_j - t_j) - t_j(z_l - t_l)) \right|_p}{\left| \sum_{j=0}^m a_{ij}z_j \sum_{l=0}^m b_{il}t_l \right|_p} = \\ &= \left| \sum_{s=0}^m (z_s - t_s) \left(\sum_{l \neq s} a_{is}b_{il} - \sum_{j \neq s} a_{ij}b_{is} \right) \right|_p \leq \frac{1}{p} \|z - t\|_p. \end{aligned} \quad (5.2)$$

As $a_{ij}, b_{il} \in \mathcal{E}_p$ then by Lemma 3 we get $a_{ij}b_{il} \in \mathcal{E}_p$. Hence, for any $i, j, l \in \{0, 1, \dots, m\}$ we have

$$|a_{ij} - b_{il}|_p = |a_{ij} - 1 + 1 - b_{il}|_p \leq \max\{|a_{ij} - 1|_p, |b_{il} - 1|_p\} < p^{-1/(p-1)} \leq \frac{1}{p}.$$

From this and by (5.2) we obtain

$$|F_i(z) - F_i(t)|_p \leq \frac{1}{p} \|z - t\|_p.$$

□

Lemma 6. *Let $\{z^{(r)}\}_{r \in \mathbb{N}}$ and $\{t^{(r)}\}_{r \in \mathbb{N}}$ are the sequences in \mathcal{E}_p^{m+1} . Then for any $n \in \mathbb{N}$ it holds*

$$\left| \prod_{s=1}^n F_i(z^{(s)}) - \prod_{s=1}^n F_i(t^{(s)}) \right|_p \leq \frac{1}{p} \max_{1 \leq s \leq n} \|z^{(s)} - t^{(s)}\|_p, \quad i = 0, 1, \dots, m. \quad (5.3)$$

Proof. We prove this by induction on n . By Lemma 5 condition (5.3) is satisfied for $n = 1$. Suppose that (5.3) is valid for n . We will prove that it is satisfied for $n + 1$. We have

$$\begin{aligned} \left| \prod_{s=1}^{n+1} F_i(z^{(s)}) - \prod_{s=1}^{n+1} F_i(t^{(s)}) \right|_p &= \left| \prod_{s=1}^n F_i(z^{(s)}) F_i(z^{(n+1)}) - \prod_{s=1}^n F_i(t^{(s)}) F_i(t^{(n+1)}) \right|_p = \\ &= \left| \left(\prod_{s=1}^n F_i(z^{(s)}) - \prod_{s=1}^n F_i(t^{(s)}) \right) F_i(z^{(n+1)}) + \left(F_i(z^{(n+1)}) - F_i(t^{(n+1)}) \right) \prod_{s=1}^n F_i(t^{(s)}) \right|_p \end{aligned} \quad (5.4)$$

By Lemma 3 and Lemma 4 we get

$$\left| F_i(z^{(n+1)}) \right|_p = 1 \quad \text{and} \quad \left| \prod_{s=1}^n F_i(t^{(s)}) \right|_p = 1.$$

Using non-Archimedean norm's property from (5.4) we obtain

$$\begin{aligned} \left| \prod_{s=1}^{n+1} F_i(z^{(s)}) - \prod_{s=1}^{n+1} F_i(t^{(s)}) \right|_p &\leq \max \left\{ \left| \prod_{s=1}^n F_i(z^{(s)}) - \prod_{s=1}^n F_i(t^{(s)}) \right|_p, \left| F_i(z^{(n+1)}) - F_i(t^{(n+1)}) \right|_p \right\} \\ &\leq \frac{1}{p} \max \left\{ \max_{1 \leq s \leq n} \|z^{(s)} - t^{(s)}\|_p, \|z^{(n+1)} - t^{(n+1)}\|_p \right\} = \frac{1}{p} \max_{1 \leq s \leq n+1} \|z^{(s)} - t^{(s)}\|_p. \end{aligned}$$

□

Consider the following functional equations

$$z_{i,x} = \prod_{y \in S(x)} F_i(z), \quad i = 0, 1, \dots, m, \quad x \in V \setminus \{x_0\}. \quad (5.5)$$

Proposition 8. *Let $m + 1$ is not divisible by p . Then (5.5) has a unique solution.*

Proof. Write

$$\mathcal{F}_i(z_x) = \prod_{y \in S(x)} F_i(z_x). \quad (5.6)$$

Let $z_x, t_x \in \mathcal{E}_p^{m+1}$ for any $x \in V \setminus \{x_0\}$. Since $|S(x)| = k$ for all $x \in V \setminus \{x_0\}$ then by Lemma 6 we have

$$|\mathcal{F}_i(z_x) - \mathcal{F}_i(t_x)|_p \leq \frac{1}{p} \max_{y \in S(x)} \|z_y - t_y\|_p \quad \text{for all } x \in V \setminus \{x_0\}. \quad (5.7)$$

Denote

$$\mathcal{F}(z) = (\mathcal{F}_0(z), \mathcal{F}_1(z), \dots, \mathcal{F}_m(z)).$$

By Lemma 4 \mathcal{F} is a function from \mathcal{E}_p^{m+1} to \mathcal{E}_p^{m+1} . From (5.7) for any $x \in V \setminus \{x_0\}$ we get

$$\|\mathcal{F}(z_x) - \mathcal{F}(t_x)\|_p \leq \frac{1}{p} \max_{y \in S(x)} \|z_y - t_y\|_p,$$

which means that the function \mathcal{F} is contractive. Then the function has a unique fixed point in \mathcal{E}_p^{m+1} . \square

Theorem 6. *Let H be a hamiltonian of $m+1$ -state p -adic SOS model on a Cayley tree Γ^k . If $p \nmid m+1$ then $|G(H)| = 1$. Moreover, a measure $\mu \in G(H)$ is a translation-invariant and symmetric.*

Proof. Let $F_m(z) \equiv 1$ and

$$F_i(z) = \frac{\sum_{j=0}^{m-1} \theta^{|i-j|_\infty} z_{j,y} + \theta^{m-i}}{\sum_{j=0}^{m-1} \theta^{m-j} z_{j,y} + 1}, \quad i = 0, 1, \dots, m-1.$$

Then by Proposition 1 and Proposition 8 there is a unique p -adic Gibbs measure for the $m+1$ -state p -adic SOS model on a Cayley tree Γ^k if $p \nmid m+1$.

Denote

$$\mathcal{I} = \{(z_0, z_1, \dots, z_m) : z_j \in \mathcal{E}_p \text{ and } z_j = z_{m-j}, j = 0, \dots, m\}.$$

It is easy to see that for any $i = 1, 2, \dots, m+1$ it holds $F_i(z) = F_{m-i}(z)$ if $z \in \mathcal{I}$. From the proof of the previous Proposition a unique solution of (5.5) belongs to \mathcal{I} . Then by Proposition 2 a unique p -adic Gibbs measure is a symmetric TI. \square

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O.N. KHAKIMOV, INSTITUTE OF MATHEMATICS, 29, DO'RMON YO'LI STR., 100125, TASHKENT, UZBEKISTAN.

E-mail address: `hakimovo@mail.ru`